

The University of Chicago

MINIMA OF DOUBLE INTEGRALS
WITH RESPECT TO UNILATERAL
VARIATIONS AND APPLICATIONS
TO SUBHARMONIC FUNCTIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1940

By

JOSEPH HARMON LEVIN

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TABLE OF CONTENTS

Section	Page
1. Introduction	1
2. Notation and preliminary remarks	3
3. Integral means	6
4. Conditions for the minimum of a double integral with respect to unilateral variations	10
5. A class of positive regular problems	27
6. Subharmonic functions	37
7. Association with the problem of Plateau in non- parametric form	48
Bibliography	51

MINIMA OF DOUBLE INTEGRALS WITH RESPECT TO UNILATERAL VARIATIONS,
AND APPLICATIONS TO SUBHARMONIC FUNCTIONS

1. Introduction. A function $z(x,y)$ having continuous second order partial derivatives in an open region R of the xy -plane is harmonic in that region if it satisfies the partial differential equation of Laplace,

$$\Delta z \equiv \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$

in R . This will be recognized as the Euler equation for the calculus of variations problem of minimizing the Dirichlet integral

$$I[z;R] \equiv \iint_R (z_x^2 + z_y^2) dx dy$$

in a class of "admissible" functions assuming specified values on the frontier R' of R . An analogous situation arises in the consideration of subharmonic functions. A function $u(x,y)$ having continuous second order partial derivatives in R is subharmonic there if and only if $\Delta u \geq 0$ in this region [11, pp. 12, 13]*.

If, in addition to satisfying these conditions, u is continuous in the closure $\bar{R}$ of R , then it is readily shown that u renders the Dirichlet integral a minimum in a class of "admissible" functions coinciding with u on R' and lying below u in R . The question arises as to whether this property is retained by subharmonic functions not having the differentiability properties

*Numbers in brackets refer to the bibliography at the end of this paper.

indicated above for u . Another question is whether this property is characteristic for subharmonic functions. These questions suggest, and will be approached by way of, the consideration of a more general problem of the calculus of variations. This more general problem is that of investigating necessary and sufficient conditions for a given surface to furnish a minimum to a double integral in a class of surfaces lying below it interior to its region of definition, but having the same values on the frontier. It is the aim of the present paper to develop such conditions for this more general problem, and to indicate how some of the defining properties of subharmonic functions may be regarded as special cases. It will be found that many of the conditions derived closely parallel the usual conditions of the calculus of variations for minima with respect to variations which are unrestricted in sign.

Section 2 of this paper is prefatory. In Section 3 are reviewed some properties of integral mean functions which are extensively employed as approximating functions in succeeding developments. Most of these are known and may be found elsewhere in the literature. Section 4 presents the development of the above mentioned necessary and sufficient conditions for minima with respect to unilateral variations. The results obtained by specializing the integrand function in various ways are sufficiently engaging as to merit particular attention in Section 5. Applications to subharmonic functions and the Dirichlet integral are made in Section 6. Some characterizing properties for subharmonic functions are there obtained as special cases of the preceding theory. Some of these results are obtained in strengthened form on account of the special nature of subharmonic functions.

Section 7 involves the application of the theory of Sections 4 and 5 to the least area problem and the problem of Plateau.

2. Notation and preliminary remarks. Throughout this paper R will be used to denote a finite open region of the xy -plane. If E is any subset of R , $\bar{E}$ will denote the closure of E , while E' will represent the set of frontier points of E .

If (x_0, y_0) is a given point, $T(x_0, y_0; h)$ will signify the open square region with side $2h$: $x_0 - h < x < x_0 + h$, $y_0 - h < y < y_0 + h$; similarly, $D(x_0, y_0; h)$ will denote the open circular region with center at (x_0, y_0) and radius h . We shall also write $Q(x_0, y_0; h)$ for $T'(x_0, y_0; h)$, and $C(x_0, y_0; h)$ in place of $D'(x_0, y_0; h)$.

Integration is to be in the sense of Lebesgue. The class of all measurable functions $g(x, y)$ for which $|g|^n$ is summable in the region of definition of g will be represented, as is customary, by L^n . We shall also designate by L^∞ the class of measurable functions $g(x, y)$ essentially bounded in their regions of definition; i.e., for each $g(x, y)$ in L^∞ there exists a constant k such that $|g| < k$ almost everywhere in the region of definition of g . If $g(x, y)$ is measurable in R , then we write for brevity,

$$M_n[g; R] \equiv \left[\int_R |g|^n dx dy \right]^{1/n},$$

$$M[g; R] \equiv M_1[g; R],$$

$$M_\infty[g; R] \equiv \text{the essential upper bound of } g \text{ in } R.$$

Then, if $m^{-1} + n^{-1} = 1$ ($m = \infty$ when $n = 1$), and if f, g are in L^m, L^n , respectively, we have the Hölder inequality,

$$M[fg; R] \leq M_m[f; R] M_n[g; R].$$

In the following pages the symbols m and n will always be used, as above, to denote positive numbers having the property that $m^{-1} + n^{-1} = 1$ ($m = \infty$ when $n = 1$). Then clearly, $m \geq 1$, $n \geq 1$.

Two measurable functions will be called equivalent in case they coincide except perhaps on a set of measure zero.

If V is an arbitrary condition or property, $E[V]$ signifies the set of all points (x,y) satisfying this condition or having this property. Thus, for example, $E[(x,y) \text{ in } R; y = y_0]$ denotes the set of points (x,y) in R for which $y = y_0$; i.e., the intersection of the line $y = y_0$ with R .

A function $f(x,y)$ is said to be of class $C^{(n)}$ in R if it is continuous in R together with its partial derivatives of the first n orders.

Consider the closed rectangle K : $a \leq x \leq b$, $c \leq y \leq d$. For a fixed y_0 in the interval (c,d) we shall denote by $V_a^b f(x,y_0)$ the total variation of $f(x,y_0)$ regarded as a function of x alone on the interval (a,b) . A corresponding meaning is attached to the symbol $V_c^d f(x_0,y)$.

DEFINITION 2:1. A continuous function $f(x,y)$ is absolutely continuous in the sense of Tonelli (abbreviated A.C.T.) on the rectangle K if

(i) it is an absolutely continuous function of x for almost every value of y on the interval (c,d) and an absolutely continuous function of y for almost every value of x on the interval (a,b) ;

(ii) $V_c^d f(x_0,y)$ and $V_a^b f(x,y_0)$ are summable functions of x_0 and y_0 on $a \leq x_0 \leq b$ and $c \leq y_0 \leq d$, respectively.

It may be readily verified that $V_c^d f(x_0,y)$ and $V_a^b f(x,y_0)$ are lower semi-continuous functions of x_0 and y_0 , respectively, on the respective intervals (a,b) and (c,d) [14, p. 169].

DEFINITION 2:2. A continuous function $f(x,y)$ is A.C.T. in a region R (or $\bar{R}$) in case it is A.C.T. on every closed interior rectangle.

DEFINITION 2:3. A function $f(x,y)$ is Lipschitzian with constant A on a given point set in case for every pair of points (x_1, y_1) and (x_2, y_2) in the set,

$$|f(x_1, y_1) - f(x_2, y_2)| \leq A[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2}.$$

Throughout the following presentation, we shall understand by the statement that a function $f(x,y)$ is absolutely continuous on an open linear set, or on the closure of an open linear set, that f is absolutely continuous on every closed interval contained in the linear set.

DEFINITION 2:4. A function $f(x,y)$ defined in R will be called absolutely continuous by sections (abbreviated a.c.s.) in R if

(i) $f(x,y)$ is continuous in R ;

(ii) $f(x,y)$ is an absolutely continuous function of x on the linear set $E[(x,y) \text{ in } R; y = y_0]$ for almost all y_0 ;

(iii) $f(x,y)$ is an absolutely continuous function of y on the linear set $E[(x,y) \text{ in } R; x = x_0]$ for almost all x_0 .

A function $f(x,y)$ satisfying in R the conditions of any one of the Definitions 2:2, 2:3, or 2:4 has first order partial derivatives which exist almost everywhere in R and are measurable in R . In particular, if $f(x,y)$ is Lipschitzian with constant A in R these partial derivatives are bounded by the Lipschitz constant A ; moreover, $f(x,y)$ is A.C.T. in R .

3. Integral means. Let $f(x,y)$ be a function summable (whose Lebesgue integral exists and is finite) in an open region R of the xy -plane, and let S be an open region contained with its frontier S' in R . Finally, let h be a positive number so small that for $|a| \leq h$, $|b| \leq h$, and (x,y) in S , the point $(x+a, y+b)$ is in R . Under these conditions the quantity

$$f^{(h)}(x,y) = \frac{1}{4h^2} \int_{-h}^h \int_{-h}^h f(x+a, y+b) da db,$$

called the integral mean of f on the square $T(x,y;h)$, is well defined in S . A summary of properties of integral means to be employed in the developments of subsequent sections will now be presented. Most of these properties are well known and will be given without proof.

PROPERTY 3:1. The function $f^{(h)}(x,y)$ is continuous in S , absolutely continuous in x for every y , and absolutely continuous in y for every x .

PROPERTY 3:2. If $f(x,y)$ is continuous in R , then $f^{(h)}(x,y)$ approaches $f(x,y)$ uniformly on S as h approaches zero. If f is only summable, then $f^{(h)}$ approaches f almost everywhere on S .

PROPERTY 3:3. Regarded as a function of four variables (x,y,a,b) , the function $f(x+a, y+b)$ is summable in the region $E[(x,y)$ in S ; $|a| \leq h$, $|b| \leq h]$. Hence, by Fubini's theorem,

$$\begin{aligned} \iint_S f^{(h)}(x,y) dx dy &= \frac{1}{4h^2} \iint_S \left\{ \int_{-h}^h \int_{-h}^h f(x+a, y+b) da db \right\} dx dy \\ &= \frac{1}{4h^2} \int_{-h}^h \int_{-h}^h \left\{ \iint_S f(x+a, y+b) dx dy \right\} da db. \end{aligned}$$

PROPERTY 3:4. Let $k(x,y)$ be a function belonging to the Lebesgue class L^m ($1 \leq m \leq \infty$) in R . If $f(x,y)$ belongs to L^n , then $k(x,y)f^{(h)}(x,y)$ converges in the mean to $k(x,y)f(x,y)$ in S as h approaches zero.

This statement is a consequence of Properties 3:1 and 3:2. To show this, let X be an arbitrary measurable set contained in R . Then, by Hölder's inequality,

$$\begin{aligned} M[k(x,y)(f(x,y) - f(x+a, y+b)); X] \\ \leq M[k(x,y)f(x,y); X] + M_m[k(x,y); X] M_n[f(x+a, y+b); X] \\ \leq M[kf; X] + M_m[k; X] M_n[f; X_0] \end{aligned}$$

where the set X_0 appearing in the final expression is merely a translation of X , and is contained in R . In view of the absolute continuity of the indefinite integral, the three integrals on the right in the above inequality tend to zero as $\text{meas.}(X)$, the measure of X , approaches zero.

Now by Property 3:3,

$$\begin{aligned} M[k(f - f^{(h)}); X] &\leq \frac{1}{4h^2} \int_X \int \left\{ \int_{-h}^h \int_{-h}^h |k(x,y)| |f(x,y) - f(x+a, y+b)| da db \right\} dx dy \\ (3:1) \quad &\leq \frac{1}{4h^2} \int_{-h}^h \int_{-h}^h M[k(x,y)(f(x,y) - f(x+a, y+b)); X] da db. \end{aligned}$$

From the foregoing paragraph, it is seen that to every $\epsilon > 0$, there corresponds a δ_ϵ such that whenever $X \subset S$, and $\text{meas.}(X) < \delta_\epsilon$, it follows that

$$M[k(x,y)(f(x,y) - f(x+a, y+b)); X] < \epsilon.$$

Consequently, for such sets X the right-hand member of (3:1) is also less than ϵ , and by this inequality the class of integrals $\left\{ \iint |k(f - f^{(h)})| dx dy \right\}$ is seen to be uniformly absolutely continuous [5, vol. 2, p. 296] on S . Moreover, $\lim_{h \rightarrow 0} |f - f^{(h)}| = 0$ almost everywhere in S by Property 3:2. Therefore [5, vol. 2, p. 297],

$$\lim_{h \rightarrow 0} M[k(f - f^{(h)}); S] = M[\lim_{h \rightarrow 0} k(f - f^{(h)}); S] = 0.$$

PROPERTY 3:5. If $f(x,y)$ is summable in R , then $f^{(h)}$ converges in the mean to $f(x,y)$ in S .

This follows at once from Property 3:4 by taking $k(x,y) \equiv 1$, $m = \infty$, $n = 1$.

PROPERTY 3:6. If $f(x,y)$ is summable in R , then almost everywhere on any linear set $E[(x,y) \text{ in } S; y = \text{const.}]$ the partial derivative $\partial f^{(h)} / \partial x$ exists and is given by

$$\frac{\partial}{\partial x} f^{(h)} = \frac{1}{4h^2} \int_{-h}^h (f(x+h, y+b) - f(x-h, y+b)) db = \frac{1}{4h^2} \int_{Q(x,y;h)} f \, dy,$$

where the latter symbol signifies a line integral. Similarly, almost everywhere on $E[(x,y) \text{ in } S; x = \text{const.}]$ the partial derivative $\partial f^{(h)} / \partial y$ exists and is given by

$$\frac{\partial}{\partial y} f^{(h)} = \frac{1}{4h^2} \int_{-h}^h (f(x+a, y+h) - f(x+a, y-h)) da = - \frac{1}{4h^2} \int_{Q(x,y;h)} f \, dx.$$

In particular, these equalities hold almost everywhere in S .

PROPERTY 3:7. Suppose that the function $f(x,y)$ is absolutely continuous in x for almost every y and that $\partial f / \partial x$ is summable on arbitrary closed subsets of R . Then

$$(3:2) \quad \frac{\partial}{\partial x} f^{(h)} = \left(\frac{\partial f}{\partial x} \right)^{(h)}$$

in S . In like manner, if $f(x,y)$ is absolutely continuous in y for almost every x , and if $\partial f / \partial y$ is summable on arbitrary closed subsets of R , then

$$(3:3) \quad \frac{\partial}{\partial y} f^{(h)} = \left(\frac{\partial f}{\partial y} \right)^{(h)}$$

in S .

It follows immediately from this proposition that under the circumstances described $f^{(h)}$ has continuous first partial derivatives which possess all the properties of integral means presented in this section. To verify (3:2), it is first noted

that Property 3:6 implies the validity of the equality almost everywhere on any linear set $E[(x,y) \text{ in } S; y = y_0]$, and hence, also, almost everywhere in S . Combining this fact with the absolute continuity of $f^{(h)}$ as a function of x for every y , and expressing $f^{(h)}$ as an indefinite integral, we have for any fixed $y = y_0$ and x_0 such that the interval (x_0, x) is in $E[(x,y) \text{ in } S; y = y_0]$,

$$f^{(h)}(x, y_0) - f^{(h)}(x_0, y_0) = \int_{x_0}^x \frac{\partial f^{(h)}}{\partial x}(x, y_0) dx = \int_{x_0}^x \left(\frac{\partial f(x, y_0)}{\partial x} \right)^{(h)} dx.$$

The integrand of the last integral is continuous so that (3:2) holds everywhere on $E[(x,y) \text{ in } S; y = y_0]$. But y_0 is arbitrary, so that (3:2) is actually valid everywhere in S . Relation (3:3) may be established by a similar argument.

PROPERTY 3:8. If $f(x,y)$ is summable in R , then $f^{(h)}(x,y)$ is A.C.T. in its region of definition.

Without loss of generality, we may restrict our attention to a rectangle $K: s \leq x \leq t, u \leq y \leq v$. Suppose that h_0 is a positive number chosen so that $(x+a, y+b)$ is in R whenever (x,y) is in K and $|a| \leq h_0, |b| \leq h_0$. For a fixed positive h such that $h \leq h_0$ it follows by Property 3:6 that almost everywhere on a set $E[(x,y) \text{ in } K; y = y_0]$ we have

$$\begin{aligned} \left| -\frac{\partial}{\partial x} f^{(h)}(x, y_0) \right| &= \left| \frac{1}{4h^2} \int_{-h}^h (f(x+h, y_0+b) - f(x-h, y_0+b)) db \right| \\ (3:4) \quad &\leq \frac{1}{4h^2} \int_{-h_0}^{h_0} |f(x+h, y_0+b) - f(x-h, y_0+b)| db. \end{aligned}$$

For such a value h , and for any y_0 in (u,v) , the point $(x+h, y_0+b)$ is in R whenever x is in (s,t) and $|b| \leq h_0$. Consequently,

$$(3:5) \quad J(h, y_0; h_0) \equiv \frac{1}{4h^2} \int_s^t \int_{-h_0}^{h_0} |f(x+h, y_0+b) - f(x-h, y_0+b)| db dx$$

exists and is finite. Furthermore [5, vol. 1, p. 605],

$$(3:6) \quad V_s^t f^{(h)}(x, y_0) = \int_s^t \left| -\frac{\partial}{\partial x} f^{(h)}(x, y_0) \right| dx, \quad (h \leq h_0, y_0 \text{ in } (u, v)).$$

By (3:4) and (3:5) we thus obtain

$$V_s^t f^{(h)}(x, y_0) \leq J(h, y_0; h_0), \quad (h \leq h_0, y_0 \text{ in } (u, v)).$$

Now $J(h, y_0; h_0)$ is an absolutely continuous function of y_0 on (u, v) for every $h \leq h_0$. This is established by the same argument as that used in the proof of the absolute continuity of $f^{(h)}(x, y)$. Thus $J(h, y_0; h_0)$, considered as a function of y_0 , is summable on (u, v) . Therefore $V_s^t f^{(h)}(x, y_0)$ is a lower semi-continuous function of y_0 on (u, v) , dominated by a summable function, and hence is itself summable. In a precisely similar fashion we may show the summability of $V_u^v f^{(h)}(x_0, y)$ with respect to x_0 on the interval (s, t) . In accordance with Definitions 2:1 and 2:2 it now follows that $f^{(h)}(x, y)$ is A.C.T. in its region of definition. By (3:6), the summability of $V_u^v f^{(h)}(x_0, y)$ and of $V_s^t f^{(h)}(x, y_0)$ with respect to x_0 and y_0 , respectively, implies the summability of $\partial f^{(h)} / \partial x$ and of $\partial f^{(h)} / \partial y$. Hence we have the following result.

PROPERTY 3:9. If $f(x, y)$ is summable in R , then $\partial f^{(h)} / \partial x$ and $\partial f^{(h)} / \partial y$ are summable on arbitrary closed sets interior to their regions of definition.

4. Conditions for the minimum of a double integral with respect to unilateral variations. The early portions of this section will be devoted to establishing some preliminary propositions which will be used in the proofs of later theorems.

LEMMA 4:1. If $g(x, y)$ is summable in R , and

$$\int_R t(x, y) g(x, y) dx dy \geq 0$$

for every function $t(x,y)$ that is Lipschitzian and non-positive on $\bar{R}$, and which vanishes on R' , then $g(x,y) \leq 0$ almost everywhere in R .

This lemma is stated for Lipschitzian functions only for direct application to the proofs of Lemma 4:2 and Theorem 4:2. The following proof would apply almost verbatim if the functions $t(x,y)$ for which the inequality of the lemma is supposed to hold were restricted to have derivatives of arbitrarily high order.

Let $C(x_0, y_0; r)$ be a circle lying with its interior in R , and let d be a constant such that for $|a| \leq d$, $|b| \leq d$, the point $(x+a, y+b)$ is in R whenever (x,y) is in $D(x_0, y_0; r)$. Now select any Lipschitzian function $t(x,y)$ which vanishes identically on the complement of $D(x_0, y_0; r)$ and is non-positive in $D(x_0, y_0; r)$. For a fixed pair (a,b) satisfying $|a| \leq d$, $|b| \leq d$, define $t_1(x,y)$ by $t_1(x+a, y+b) = t(x,y)$. Then

$$(4:1) \iint_{D(x_0, y_0; r)} t(x,y)g(x+a, y+b)dx dy = \iint_{D_1} t_1(x,y)g(x,y)dx dy \geq 0,$$

where $D_1 = D(x_0+a, y_0+b; r)$, in view of the hypothesis of the lemma. Integrating with respect to a and b , we thus obtain,

$$\frac{1}{4h^2} \int_{-h}^h \int_{-h}^h \left\{ \iint_{D(x_0, y_0; r)} t(x,y)g(x+a, y+b)dx dy \right\} da db \geq 0, \quad (0 < h \leq d).$$

By Fubini's theorem, we may interchange the order of integration and conclude that

$$(4:2) \iint_{D(x_0, y_0; r)} t(x,y)g^{(h)}(x,y)dx dy \geq 0.$$

But $g^{(h)}(x,y)$ is continuous in $D(x_0, y_0; r)$ and (4:2) is valid for all $t(x,y)$ of the type described. Hence, it is easily seen that $g^{(h)}(x,y) \leq 0$ in $D(x_0, y_0; r)$ for every h satisfying the condition $h \leq d$. Therefore, $g(x,y) \leq 0$ almost everywhere in $D(x_0, y_0; r)$, by Property 3:2. And now, since R may be expressed

as the sum, in the point set sense, of a denumerable number of circles, the desired conclusion is established.

Corresponding to any open set S comprised in R , $\mathcal{N}_S^m$ shall represent the class of functions $\eta(x,y)$ which are continuous in $\bar{R}$, a.c.s. in R and have the further properties:

(i) η_x, η_y belong to the class L^m ;

(ii) η vanishes on the set $\bar{R} - S$.

For simplicity of notation we shall write $\mathcal{N}^m$ in place of $\mathcal{N}_R^m$. It follows at once from this definition that if $\eta(x,y)$ is in the class $\mathcal{N}_S^m$, then $\eta(x,y)$ is A.C.T. in $\bar{R}$. It is a further consequence of this definition that if $\eta(x,y)$ is a non-positive function belonging to the class $\mathcal{N}_S^m$, then almost everywhere in $R - S$ the derivatives η_x and η_y exist and are equal to zero. Since these derivatives are clearly zero on $\bar{R} - S$, it is only necessary to show that $\eta_x = \eta_y = 0$ almost everywhere on S' . Now, since η is A.C.T. in $\bar{R}$, the partial derivatives η_x, η_y exist almost everywhere in $\bar{R}$, and hence they exist almost everywhere on S' . But η attains a maximum at each point of S' since it is a non-positive function vanishing on S' , so that almost everywhere on S' we have $\eta_x = \eta_y = 0$.

We next introduce a notion of "almost all squares" in a given region.

DEFINITION 4:1. Let $\mathcal{E}$ be a class of squares contained with their interiors in an open region R . Corresponding to a given positive h , let N_h denote the set of all points (x,y) in R such that $\bar{T}(x,y;h)$ is contained in R . If for each h the set of points (x,y) of R for which $\bar{T}(x,y;h)$ is in N_h but not in $\mathcal{E}$ has measure zero, then $\mathcal{E}$ is said to constitute almost all squares of R .

LEMMA 4:2. Suppose that $g_1(x,y)$ is summable in R , that $g_2(x,y)$, $g_3(x,y)$ are functions belonging to the class L^m in R , and that for every non-positive $\eta(x,y)$ in the class $\mathcal{N}^n$ the inequality

$$(4:3) \quad \iint_R (\eta g_1 + \eta_x g_2 + \eta_y g_3) dx dy \geq 0$$

holds. Then, for almost all squares Q of R ,

$$(4:4) \quad \int_Q g_2 dy - g_3 dx \geq \int_T g_1 dx dy.$$

Conversely, if g_1 is summable in R , if g_2, g_3 belong to the class L^m in R , and if (4:4) holds for almost all squares of R , then (4:3) is true for every non-positive η in $\mathcal{N}^n$.

To verify the first part of this lemma, it is sufficient to confine our attention to Lipschitzian η 's. Choose any open set S contained with its frontier S' in R . Let d be a positive number such that if $|a| \leq d$, $|b| \leq d$, then $(x+a, y+b)$ is in R whenever (x,y) is in S . For such pairs (a,b) , and for Lipschitzian η 's vanishing on S' it follows, by the same reasoning which led to (4:1) in the proof of Lemma 4:1, that

$$\begin{aligned} \iint_S [\eta(x,y)g_1(x+a,y+b) + \eta_x(x,y)g_2(x+a,y+b) \\ + \eta_y(x,y)g_3(x+a,y+b)] dx dy \geq 0. \end{aligned}$$

Integrating this expression with respect to a and b between the limits $(-h,h)$, where $0 < h \leq d$, and interchanging the order of integration, we conclude that

$$(4:5) \quad \iint_S (\eta g_1^{(h)} + \eta_x g_2^{(h)} + \eta_y g_3^{(h)}) dx dy \geq 0, \quad (0 < h \leq d).$$

Now, in view of Property 3:1, the functions $\eta g_2^{(h)}$ and $\eta g_3^{(h)}$ are a.c.s. and almost everywhere in S their derivatives are given

by the usual formula for the derivative of a product. The functions $\eta_x g_2^{(h)}$ and $\eta_y g_3^{(h)}$ are clearly summable in S . Furthermore, since η is bounded in S , the summability of $\eta \partial g_2^{(h)} / \partial x$ and $\eta \partial g_3^{(h)} / \partial y$ is implied by that of $\partial g_2^{(h)} / \partial x$ and of $\partial g_3^{(h)} / \partial y$ [Property 3:9]. It therefore follows that the partial derivatives $\partial(\eta g_2^{(h)}) / \partial x$ and $\partial(\eta g_3^{(h)}) / \partial y$ are summable in S . Inequality (4:5) may accordingly be rewritten as follows:

$$(4:6) \quad \iint_S \eta (g_1^{(h)} - \frac{\partial}{\partial x} g_2^{(h)} - \frac{\partial}{\partial y} g_3^{(h)}) dx dy + \iint_S [\frac{\partial}{\partial x} (\eta g_2^{(h)}) + \frac{\partial}{\partial y} (\eta g_3^{(h)})] dx dy \geq 0, \quad (0 < h \leq d).$$

Now $\eta g_2^{(h)}$ and $\eta g_3^{(h)}$ are a.c.s. in S , and vanish on S' since η vanishes there. Consequently, the second integral in (4:6) is zero, as can be easily seen if we first express it as an iterated integral, and then apply the fundamental theorem of the integral calculus for the Lebesgue integral. Hence

$$\iint_S \eta (g_1^{(h)} - \frac{\partial}{\partial x} g_2^{(h)} - \frac{\partial}{\partial y} g_3^{(h)}) dx dy \geq 0, \quad (0 < h \leq d),$$

for every non-positive, Lipschitzian η vanishing on S' . And now, applying Lemma 4:1, it results that

$$(4:7) \quad g_1^{(h)} - \frac{\partial}{\partial x} g_2^{(h)} - \frac{\partial}{\partial y} g_3^{(h)} \leq 0, \quad (0 < h \leq d),$$

almost everywhere in S .

The set of points N_h appearing in Definition 4:1 is open. Therefore, for each h the inequality (4:7) holds by the foregoing proof almost everywhere in N_h . But by the definition of integral mean, and by Property 3:6, this is equivalent to the statement that for a fixed h , $0 < h \leq d$ the inequality

$$(4:8) \quad \int_{Q(x,y;h)} g_2 dy - g_3 dx \geq \iint_{T(x,y;h)} g_1 dx dy$$

holds almost everywhere in N_n . Then, by Definition 4:1, the class of squares for which (4:8) holds constitutes almost all squares of R .

The validity of the converse statement will be established in two steps. Under the assumption that (4:4) holds for almost all squares in R , we shall first prove that the inequality (4:3) is true when η is in the class $\mathcal{N}_S^n$ and S is contained with its frontier in R . By Property 3:4,

$$(4:9) \quad \lim_{h \rightarrow 0} \iint_S (\eta g_1^{(h)} + \eta_x g_2^{(h)} + \eta_y g_3^{(h)}) dx dy \\ = \iint_S (\eta g_1 + \eta_x g_2 + \eta_y g_3) dx dy.$$

Again, the left-hand member may be written

$$(4:10) \quad \lim_{h \rightarrow 0} \left\{ \iint_S \eta (g_1^{(h)} - \frac{\partial}{\partial x} g_2^{(h)} - \frac{\partial}{\partial y} g_3^{(h)}) dx dy \right. \\ \left. + \iint_S [\frac{\partial}{\partial x} (\eta g_2^{(h)}) + \frac{\partial}{\partial y} (\eta g_3^{(h)})] dx dy \right\}.$$

The second term vanishes as in (4:6). Owing to the equivalence of inequalities (4:7) and (4:8), and in view of the non-positiveness of η , the integrand in the first term of (4:10) is non-negative, and therefore the right-hand member of (4:9) is non-negative.

But $\eta \equiv 0$ in $R - S$, and by the remarks following the definition of the class $\mathcal{N}_S^m$ it is seen that $\eta_x = \eta_y = 0$ almost everywhere in $R - S$. Consequently, inequality (4:3) holds.

Suppose, secondly, that S' coincides wholly or in part with R' . As before, we propose to establish inequality (4:3), where η is a non-positive function in the class $\mathcal{N}_S^n$. If, for $i = 1, 2, \dots$, we let $S_1 = E[(x, y) \text{ in } S; \eta(x, y) < -1/i]$, and $S_0 = E[(x, y) \text{ in } S; \eta(x, y) < 0]$, then it is seen that $S - S_0 = E[(x, y) \text{ in } S; \eta(x, y) = 0]$, and $S_0 = \sum_1 S_1$. Furthermore,

$S_1 \subset S_{1+1}$ for each i , and $\lim_{i \rightarrow \infty} \text{meas}(S_0 - S_1) = 0$. Now define functions η_1 as follows:

$$(4:11) \quad \begin{aligned} \eta_1(x, y) &= \eta + \frac{1}{i} \quad \text{in } S_1, \\ \eta_1(x, y) &\equiv 0 \quad \text{in } R - S_1. \end{aligned}$$

Evidently, $\eta_1(x, y) = 0$ on S_1' , and $\eta_1(x, y) < 0$ in S_1 . It is also apparent that η_1 is a.c.s. in R . The function η_1 belongs to the class $\mathcal{N}_{S_1}^n$ and hence, again by the discussion following the definition of the class $\mathcal{N}_S^m$, we conclude that $\eta_{1x} = \eta_{1y} = 0$ almost everywhere in $R - S_1$. Moreover, $\eta_{1x} = \eta_x$ and $\eta_{1y} = \eta_y$ wherever η_x, η_y exist in S_1 . These remarks, together with (4:11), enable us to write

$$(4:12) \quad \begin{aligned} &\iint_R (\eta_1 g_1 + \eta_{1x} g_2 + \eta_{1y} g_3) dx dy \\ &= \frac{1}{i} \iint_{S_1} g_1 dx dy + \iint_{S_0} (\eta g_1 + \eta_x g_2 + \eta_y g_3) dx dy \\ &\quad - \iint_{S_0 - S_1} (\eta g_1 + \eta_x g_2 + \eta_y g_3) dx dy. \end{aligned}$$

Now the first integral of the right-hand member of (4:12) tends to zero as $i \rightarrow \infty$ since g_1 is summable in S_0 ; the limit of the third integral is also zero because $\lim_{i \rightarrow \infty} \text{meas.}(S_0 - S_1) = 0$. Furthermore, η belongs to $\mathcal{N}_{S_0}^n$ by definition of the set S_0 , and hence $\eta \equiv 0$ and $\eta_x = \eta_y = 0$ almost everywhere in $R - S_0$. Therefore, the second integral of the right-hand member of (4:12) is equal to the corresponding integral over R . It is possible of course for S_0 to be identical with R . We thus conclude, finally, that the limit of the integral of the left-hand side of (4:12) exists as $i \rightarrow \infty$ and is equal to the integral occurring in inequality (4:3). But each function $\eta_1(x, y)$ vanishes identically in a neighborhood of S_0' , and so by the first part of the proof the

left-hand member of (4:12) is non-negative for each i . This completes the proof.

By strengthening the hypotheses on g_1, g_2, g_3 in Lemma 4:2, the following important result may be deduced.

COROLLARY. If in Lemma 4:2 the function g_1 is continuous while g_2, g_3 are of class C^1 in R , then inequality (4:3) implies that

$$\frac{\partial}{\partial x} g_2 + \frac{\partial}{\partial y} g_3 - g_1 \geq 0$$

in R . Conversely, the validity of this inequality implies (4:3) for every non-positive η in $\mathcal{R}^n$.

The proof involves integration by parts and is basically like that of Lemma 4:2. The procedure, however, is now more direct, for it is no longer necessary to use integral means as approximating functions.

The foregoing results will now be applied in obtaining some of the key theorems of this paper. Let R be a bounded open set and let $f(x, y, z, p, q)$ be a function defined and continuous for (x, y) in $\bar{R}$, and for (z, p, q) in an open region D of 3-dimensional space, not necessarily finite. The function f will be assumed to be of class $C^{(a)}$ in its last three arguments throughout D .

Presently a class of "admissible" functions, or surfaces, will be introduced. We shall be interested in establishing certain conditions that a specific admissible surface $E: z = Z(x, y)$ furnish a relative minimum to the integral

$$I \equiv \iint_R f(x, y, z(x, y), z_x(x, y), z_y(x, y)) dx dy$$

in a prescribed class of surfaces $z(x, y)$ which are dominated by $Z(x, y)$ and coincide with $Z(x, y)$ on the frontier of R . Obviously, we could just as well develop conditions that an admissible surface

Z furnish a relative minimum to the integral I in a corresponding class of surfaces dominating Z and coinciding with Z on R' . Conditions for this latter problem would be identical with those for the former except for changes in direction of certain inequalities. Hence, for simplicity, whenever there is occasion to refer to a condition of the latter sort, we shall give the reference to the corresponding condition for the problem actually under consideration.

Throughout this section, as well as subsequent sections, it will be understood, unless otherwise stated, that whenever we write, f , f_z , f_p , etc., the arguments to be assumed are $(x, y, Z(x, y), Z_x(x, y), Z_y(x, y))$.

DEFINITION 4:2. A function $z(x, y)$ defined in $\bar{R}$ will be termed admissible of class (m) , or simply admissible (m) in case:

- (i) it is single valued and measurable in $\bar{R}$;
- (ii) $z_x(x, y)$, $z_y(x, y)$ exist almost everywhere in R and are measurable in R ;
- (iii) for each (x, y) in R the point $(z(x, y), z_x(x, y), z_y(x, y))$ is in D ;
- (iv) the function $f_z(x, y, z, z_x, z_y)$ is summable in R , while f_p, f_q with the same arguments belong to the class L^m ;
- (v) the function $f(x, y, z, z_x, z_y)$ is summable in R .

It is to be noted that the first three conditions imply measurability of f , f_z , f_p , f_q with arguments (x, y, z, z_x, z_y) , and that the admissibility of z depends on the function f .

On account of (iv) in the above definition, we deduce as a special case of Lemma 4:2 the following theorem, which corresponds to the so-called Fundamental Lemma of the calculus of variations.

THEOREM 4:1. Suppose that E: $z = Z(x,y)$ is admissible (m) and that for every non-positive $\eta(x,y)$ in the class $\mathcal{N}^n$ the inequality

$$(4:13) \quad I_1[\eta] \equiv \iint_R (\eta f_z + \eta_x f_p + \eta_y f_q) dx dy \geq 0$$

holds. Then, for almost all squares Q in R,

$$(4:14) \quad \int_Q f_p dy - f_q dx \geq \iint_T f_z dx dy.$$

Conversely, if for some Z admissible (m) the relation (4:14) holds for almost all squares in R, then (4:13) is true for every non-positive function η in $\mathcal{N}^n$.

The expression appearing in (4:13) is of the form of the first variation of a double integral variation problem. The inequality (4:14) will be recognized as analogous to the integral form of the Haar differential equation for a double integral variation problem, and will frequently be referred to as the Haar integral inequality, or merely the Haar inequality.

If $Z(x,y)$ is of class $C^{(2)}$ in R, then f_z and the partial derivatives of f_p, f_q with respect to x and y are continuous in this region. Hence, by the corollary to Lemma 4:2, we deduce the following proposition.

COROLLARY. If, in addition to satisfying the conditions of Theorem 4:1, the function $Z(x,y)$ is of class $C^{(2)}$ in R, then the non-negativeness of the first variation implies that

$$\frac{\partial}{\partial x} f_p + \frac{\partial}{\partial y} f_q - f_z \geq 0$$

in R, and conversely.

The foregoing differential expression is the familiar Euler expression for a double integral variation problem. It should be recognized, in the following work, that whenever the

function $Z(x,y)$ is of class $C^{(2)}$, then the above Euler inequality may be used in place of the Haar integral inequality.

We are now prepared to present some necessary conditions and some sufficient conditions that an admissible (m) function $Z(x,y)$ provide a relative minimum to the integral I with respect to unilateral variations. We shall concern ourselves with strong relative minima.

DEFINITION 4:3. A function $\eta(x,y)$ is an admissible variation of class m or more simply, an admissible variation (m) in case:

- (i) η belongs to the class $\mathcal{N}^n$, where, it will be recalled, $n^{-1} = 1 - m^{-1}$;
- (ii) if $z(x,y)$ is an admissible surface (m), then for a sufficiently small in absolute value the point $(z + a\eta, z_x + a\eta_x, z_y + a\eta_y)$ is in D for (x,y) in R , and the function $f(x,y,z + a\eta, z_x + a\eta_x, z_y + a\eta_y)$ is summable in R .

In particular, if η belongs to $\mathcal{N}_S^n$, where S is an open subset of R , η will be called an admissible variation relative to S . Given a $Z(x,y)$ admissible (m), we shall for convenience sometimes employ the phrase "corresponding admissible variation" to denote an admissible variation (m) associated with Z . Of particular interest is the case $m = n = 2$.

It will be noted that z is not required to be continuous in Definition 4:2. On the other hand, admissible variations must be continuous on account of (i) of Definition 4:3.

It is important to recognize that if Z is admissible (m) and η is a corresponding admissible variation, then it is not a consequence of our definitions that $Z + a\eta$ is an admissible surface (m) since in general condition (iv) of Definition 4:2 may not be satisfied along $Z + a\eta$. Although this latter property is

in general missing, it seems more convenient for our present considerations to define admissible surface and admissible variation as above. Correspondingly, we shall speak of an admissible surface rendering a double integral a minimum with respect to corresponding admissible variations, rather than rendering the integral a minimum in a class of admissible surfaces.

It will be necessary for our purposes in the following paragraphs to distinguish between proper and improper minima. The admissible surface $E: z = Z(x, y)$ renders I an improper minimum in a class $\mathcal{C}$ of surfaces $z(x, y)$ in case

$$\iint_R f(x, y, Z, Z_x, Z_y) dx dy \leq \iint_R f(x, y, z, z_x, z_y) dx dy$$

for every $z(x, y)$ in $\mathcal{C}$ not equivalent to $Z(x, y)$. For a proper minimum the equality sign is excluded.

Suppose that a function $Z(x, y)$ is admissible (m). Then we shall say for brevity that $Z(x, y)$ satisfies the condition (A^m) with respect to the function f and constant $d > 0$ [or, where there is no risk of confusion, merely, Z satisfies the condition (A^m)], in case whenever $|r| < d$, $|s| < d$, $|t| < d$, then $Z + r$, $Z_x + s$, $Z_y + t$ is in D , and moreover f_z , f_p , f_q with arguments $(x, y, Z + r, Z_x + s, Z_y + t)$ are all less in absolute value than a function $M(x, y)$ summable in R , and independent of r, s, t .

If $Z(x, y)$ is admissible (m) and Z_x, Z_y belong to L^m , we might note that condition (A^m) is valid if there exist constants K_1, K_2 , and a function $M_0(x, y)$ summable in R , such that for $|r| < d$, $|s| < d$, $|t| < d$, the functions f_z, f_p, f_q with arguments $(x, y, Z + r, Z_x + s, Z_y + t)$ are all less in absolute value than

$$K_1 |f(x, y, Z, Z_x, Z_y)| + K_2 (|Z_x|^m + |Z_y|^n) + M_0(x, y).$$

For the two particular examples $f = p^2 + q^2$ and $f = (1 + p^2 + q^2)^{1/2}$ it may be shown that any function Z admissible (2) satisfies (A^2) with respect to f and any constant $d > 0$. In fact, for $f = p^2 + q^2$ one might take $K_1 + K_2 = 1$ and $M_0 = d^2 + 2$; for $f = (1 + p^2 + q^2)^{1/2}$ the values $K_1 = K_2 = 0$, $M_0 = 1$ are suitable.

THEOREM 4:2. Suppose $E: z = Z(x, y)$ is an admissible surface (m) satisfying condition (A^m) , and that E furnishes a relative minimum to the integral I with respect to non-positive admissible variations η . Then: (i) $I_1[\eta] \geq 0$; (ii) for almost all squares in R the Haar inequality (4:14) holds.

We shall commence by proving (i) for Lipschitzian functions $\eta(x, y)$. This limitation is to be removed later. By hypothesis, if a is non-negative and less than some positive constant $1/k'$ where $k' > 1$, then

$$(4:15) \quad \int_R f(x, y, Z, Z_x, Z_y) dx dy \leq \int_R f(x, y, Z + a\eta, Z_x + a\eta_x, Z_y + a\eta_y) dx dy.$$

We note that for $a = 0$, the equality holds. The partial derivative of the integrand on the right with respect to the parameter a exists and is expressed by

$$\frac{\partial}{\partial a} f(x, y, Z + a\eta, Z_x + a\eta_x, Z_y + a\eta_y) = \eta f_z + \eta_x f_p + \eta_y f_q,$$

the arguments of f_z , f_p , f_q being the same as those of f on the left. Suppose that k'' is an upper bound for $|\eta|$, and that k''' is the Lipschitzian constant corresponding to η . Then let $k = \max \{dk', k'', k'''\}$, d being the number appearing in condition (A^m) , and keep $|a| < d/k$. Then $a\eta$, $a\eta_x$, $a\eta_y$ are all less in absolute value than d . As a consequence of condition (A^m) we therefore conclude that

$$\left| \frac{\partial}{\partial a} f(x, y, Z + a\eta, Z_x + a\eta_x, Z_y + a\eta_y) \right| \leq 3dM(x, y), \quad (|a| < d/k)$$

in S . One may accordingly differentiate in (4:15) under the integral sign whenever $|a| < d/k$. In particular, inasmuch as the equality holds in (4:15) when $a = 0$, we have $I_1[\eta] \geq 0$, which is the desired conclusion (i) for Lipschitzian η 's. The reader will remember at this juncture that the first part of Theorem 4:1 was proved merely under the assumption that η satisfy a Lipschitz condition. Statement (ii) of the present theorem is a direct consequence of that result. But again, by Theorem 4:1, it follows that statement (ii) implies (i) with the restriction on the character of η removed.

In the foregoing two theorems it was not necessary to assume the function $Z(x,y)$ continuous. However, continuity of Z is essential in the proof of the following result.

THEOREM 4:3. Suppose that $E: z = Z(x,y)$ is continuous and admissible (m), and that E renders the integral I a proper relative minimum in the class of functions $z(x,y)$ such that $z = Z$ on R' , and $Z - d \leq z \leq Z + \eta \leq Z$ in R , η being an admissible variation (m). Let S be any subregion of R for which there exists a continuous function $W(x,y)$ that is admissible (m), coincides with Z on S' , and has the following properties:

(i) W renders the integral $\int_S f dx dy$ a relative minimum with respect to all variations which are admissible (m) and not greater in absolute value than d ;

(ii) $|Z - W| \leq d$ in S ;

(iii) the function defined to be equal to $Z - W$ in $\bar{S}$ and identically zero in $\bar{R} - S$ is an admissible variation (m).

Then $Z \leq W$ in S .

This theorem could be strengthened inasmuch as all the implications of admissibility are not utilized in the proof. However, the usefulness of this theorem in following sections

will be in connection with absolute minima of double integrals. It will be noted in Section 5 for example, that under circumstances there prescribed a surface which is admissible (m) and furnishes a relative minimum to the integral I , furnishes, in actuality, an absolute minimum to I . Hence, the present theorem will be more effective there if stated for admissible functions and variations. It states in essence that a function $Z(x,y)$ minimizing the integral I with respect to non-positive variations is dominated by any function $W(x,y)$ which coincides with Z on the boundary of R and which furnishes I with a relative minimum with respect to general variations, provided that $Z(x,y)$ lie close enough to $W(x,y)$.

If the theorem be false, then there must exist a point (x_0, y_0) in R at which $Z(x_0, y_0) > W(x_0, y_0)$, and hence an open subset N of R containing (x_0, y_0) in which $Z > W$ and on the frontier of which $Z = W$. Then, since $|Z - W| \leq d$ in R , we have, by virtue of (4:11) and the hypothesized minimizing property of $Z(x,y)$,

$$(4:16) \quad \iint_N f(x,y,Z,Z_x,Z_y) dx dy < \iint_N f(x,y,W,W_x,W_y) dx dy.$$

On the other hand, the minimizing property of $W(x,y)$ implies the contradictory statement,

$$(4:17) \quad \iint_N f(x,y,Z,Z_x,Z_y) dx dy \geq \iint_N f(x,y,W,W_x,W_y) dx dy.$$

Accordingly, the assumption that $Z > W$ at some point of R must be rejected. It should be noted that the conclusion of the theorem would also hold if it had been required that W furnish a proper relative minimum while Z be permitted to furnish but an improper relative minimum. The contradiction in the proof would then arise from the fact that the equality sign would be admitted in (4:16) but not in (4:17).

The Weierstrass excess function is defined to be

$$\begin{aligned} \mathcal{E}(x, y, z, p, q; P, Q) &= f(x, y, z, P, Q) - f(x, y, z, p, q) \\ &\quad - (P - p)f_p(x, y, z, p, q) - (Q - q)f_q(x, y, z, p, q). \end{aligned}$$

Suppose that admissible surfaces and variations are required to be piecewise of class C^1 . For this case L. M. Graves [2] has produced a proof of the Weierstrass condition for multiple integral problems which may be applied with almost no alteration to the unilateral case. It is to be remarked at this point that under the conditions of this paragraph, if Z is an admissible surface, and if η is an admissible variation, then $Z + a\eta$ is an admissible surface for a sufficiently small a . The Weierstrass condition is as follows:

THEOREM 4:4. If $E: z = Z(x, y)$ is piecewise of class C^1 and minimizes I with respect to unilateral variations, then at each element $(x, y, Z(x, y), Z_x(x, y), Z_y(x, y))$ of E the inequality $\mathcal{E}(x, y, Z, Z_x, Z_y; P, Q) \geq 0$ holds for every pair P, Q such that (Z, P, Q) is in D .

From this, by the usual application of Taylor's theorem we derive the Legendre condition:

THEOREM 4:5. Under the hypotheses of Theorem 4:4, at each element (x, y, Z, Z_x, Z_y) of E the quadratic form

$$(4:18) \quad A^2 f_{pp} + 2AB f_{pq} + B^2 f_{qq}$$

in (A, B) is positive semi-definite.

A sufficiency proof by an expansion method applicable to the present problem has been developed by W. T. Reid [12] in a paper to be published soon. For simplicity, Reid's notation will be closely followed in submitting the corresponding theorem for the unilateral case.

In the remainder of this section, we shall understand that all admissible functions, as well as admissible variations, are to be Lipschitzian in $\bar{R}$. Here again, it is noted that if Z is an admissible surface and if η is an admissible variation, then the surface $Z + a\eta$ is admissible for a sufficiently small. A surface $z = z(x, y)$ is called non-singular if $f_{pp}f_{qq} - f_{pq}^2 \neq 0$ at all elements $(x, y, z(x, y), z_x(x, y), z_y(x, y))$ belonging to the surface. A function $u(x, y)$ which is continuous in $\bar{R}$, of class C^1 in R , and whose partial derivatives u_x, u_y coincide with functions continuous in $\bar{R}$, will be said to satisfy the integral form of the Haar accessory equation if

$$\int_Q \omega_\pi(x, y, u, u_x, u_y) dy - \omega_\kappa(x, y, u, u_x, u_y) dx = \iint_T \omega_\tau(x, y, u, u_x, u_y) dx dy$$

holds for all squares Q of R , where

$$2\omega(x, y, z, r, \kappa) = f_{pp}\pi^2 + 2f_{pq}\pi\kappa + f_{qq}\kappa^2 + 2f_{pz}\pi r + 2f_{qz}\kappa r + f_{zz}r^2.$$

We shall say that a function $Z(x, y)$ satisfies the condition IV^* if there exists a solution of the kind described above of the Haar accessory equation which is nowhere zero in $\bar{R}$. Moreover, E is said to satisfy condition II_N^* in case there exists in (x, y, z, p, q) -space a one-sided neighborhood N of the elements (x, y, Z, Z_x, Z_y) of the form N : (x, y) in $\bar{R}$, $|p - Z_x(x, y)| \leq a_1$, $|q - Z_y(x, y)| \leq a_2$, $Z(x, y) - a_3 \leq z \leq Z(x, y)$, such that for (x, y, z, p, q) in N , and $(P, Q) \neq (p, q)$, we have $\mathcal{Q}(x, y, z, p, q; P, Q) > 0$. It is clear that the validity of condition II_N^* for E , together with the non-singularity of E , implies the strengthened form of the Legendre condition; that is, the quadratic form (4:18) is positive definite.

The proof of Reid's theorem depends on two preliminary lemmas which will not be stated here. The theorem for the unilateral case is as follows:

THEOREM 4:6. If $E: z = Z(x,y)$ is a non-singular admissible surface that is continuous in $\bar{R}$, of class C^1 in R , which satisfies the Haar inequality, and also conditions II_N^* and IV^* , then E renders I a strong relative minimum with respect to non-positive admissible variations.

5. A class of positive regular problems. As a special case of the problems treated in the foregoing section, we now turn to integrand functions $f(x,y,p,q)$ which are continuous in all four arguments, and of class $C^{(2)}$ for (x,y) in $\bar{R}$ and p, q arbitrary. The problem will be assumed to be positive regular; i.e., the quadratic form

$$(5:1) \quad f_{pp}A^2 + 2f_{pq}AB + f_{qq}B^2$$

is positive definite for all (x,y,p,q) in the region of definition of f . All the theorems of the last section may, of course, be applied to the present situation. But, owing to the special nature of the integrand function, it is natural to expect that the results to be obtained here would be somewhat more individualized. The most conspicuous phenomena to be observed in the sequel are the identically vanishing of the partial derivative f_z , and the fact that the results provide for absolute minima. The latter feature is a consequence of the hypothesis of positive regularity. Since $f_z \equiv 0$, (4:13) reduces to

$$I_1[\gamma] = \iint_R (\eta_x f_p + \eta_y f_q) dx dy \geq 0,$$

while the Haar inequality (4:14) becomes

$$\int_Q f_p dy - f_q dx \geq 0.$$

Outside of these changes, the statement of Theorem 4:1 remains

the same for the case considered here. Theorems 4:2 and 4:3 will be interpreted in the light of the hypothesis of positive regularity. Theorem 5:1, which corresponds to Theorem 4:6, is here proved without the qualification previously demanded that the surface be continuous in $\bar{R}$ and of class C' in R . Theorems 5:2 and 5:3 have no counterpart in the preceding treatment. They are proved only for functions $f(p,q)$, not involving x, y, z explicitly. In Section 6 corresponding results will be proved under weaker hypotheses for the particular case where $f(p,q) = p^2 + q^2$, the integrand of the Dirichlet integral.

By applying Taylor's theorem with remainder to the Weierstrass $\mathcal{E}$ -function for this problem, we find in view of the positive definiteness of the quadratic form (5:1) that the $\mathcal{E}$ -function is positive for $(P,Q) \neq (p,q)$ and the strengthened condition of Legendre is satisfied throughout the region under consideration. We can now derive a stronger result than Theorem 4:6. For, suppose that the Haar inequality holds for $Z(x,y)$ admissible (m), and let $z(x,y) = Z(x,y) + \eta(x,y)$ be any other surface such that η is a non-positive variation admissible (m) relative to R . Then, by Taylor's theorem with remainder,

$$\begin{aligned}
 (5:2) \quad I &\equiv \iint_R f(x,y,z_x,z_y) dx dy - \iint_R f(x,y,Z_x,Z_y) dx dy \\
 &= \iint_R (\eta_x f_p(x,y,Z_x,Z_y) + \eta_y f_q(x,y,Z_x,Z_y)) dx dy \\
 &\quad + \frac{1}{2} \iint_R (\eta_x^2 \bar{f}_{pp} + 2\eta_x \eta_y \bar{f}_{pq} + \eta_y^2 \bar{f}_{qq}) dx dy,
 \end{aligned}$$

where $\bar{f}_{pp}$, etc., signify $f_{pp}(x,y,Z_x + \theta \eta_x, Z_y + \theta \eta_y)$, etc., and $0 < \theta(x,y) < 1$. It is to be noted that the integral of the quadratic expression on the right is finite in consequence of the expansion of $f(x,y,z_x,z_y)$ by Taylor's theorem with remainder, and the summability of $f(x,y,z_x,z_y)$, $f(x,y,Z_x,Z_y)$, and $\eta_x f_p + \eta_y f_q$.

By virtue of Theorem 4:1 and the positive regularity of the problem it then follows that $\Delta I > 0$ for $\eta \neq 0$ in R . We have thus proved the following result.

THEOREM 5:1. Under the conditions imposed above, if the Haar inequality is satisfied for $Z(x,y)$ an admissible function (m) , then $Z(x,y)$ renders the integral I a proper absolute minimum with respect to corresponding non-positive admissible variations.

It results from this theorem and Theorem 4:2 that if an admissible (m) function Z furnishes a relative minimum to I then it actually furnishes an absolute minimum. Hence, for the type of multiple integrals considered in this section, Theorems 4:2 and 4:3 are not rendered weaker by supposing that Z furnishes an absolute minimum to the integral I .

A type of function employed in the discussion of surfaces minimizing an integral $\iint f(p,q)dx dy$ is the so-called saddle-function [9, pp. 62,63]. The definition of such a function depends on the following definition of monotonicity for a function of two variables.

DEFINITION 5:1. A function $g(x,y)$ continuous in $\bar{R}$ is monotonic in R in case for every connected open set S contained in R , the function $g(x,y)$ attains the maximum and the minimum of its values in $\bar{S}$ on the frontier S' .

DEFINITION 5:2. If the function $g(x,y) - (ax + by + c)$ is monotonic in R for every choice of the constants a, b, c , then $g(x,y)$ is called a saddle-function in R .

It may be shown that under prescribed conditions a continuous function minimizing the integral I with respect to unrestricted variations is a saddle-function [9, pp. 65, 66]. In our problem, we can prove that an admissible (m) function $Z(x,y)$ continuous in $\bar{R}$, a.c.s. in R , whose first order

partial derivatives belong to L^n , and which minimizes I with respect to non-positive admissible(m) variations is what might be called an upper semi-saddle function; i.e., it is a function such that for every choice of constants a, b, c , and for every open subregion S in R , the function $Z(x,y) - (ax + by + c)$ attains the maximum of its values in $\bar{S}$ on the frontier S' . For, suppose that $Z(x,y)$ is not such a function. Then there exist constants a, b, c and a region S in R such that for some point (x_0, y_0) in S we have $Z(x_0, y_0) - (ax_0 + by_0 + c) > M'$, where M' is the maximum of $Z(x,y) - (ax + by + c)$ on S' . Hence, letting $z(x,y) = ax + by + c + M'$, there exists a connected open subset N of S in which $z(x,y) - Z(x,y) < 0$ and on the frontier of which $z - Z = 0$. Moreover, in view of our hypotheses on Z , the function $z - Z$ is an admissible variation (m) relative to N . Now, it is easily shown that the plane $z(x,y)$ renders the integral I an absolute minimum with respect to unrestricted variations, and hence, by Theorem 4:3 and the remarks made in this section on that theorem, we conclude that $Z(x,y) \leq ax + by + c + M'$ in N . This, however, contradicts our assumption on the character of Z . Geometrically, the property we have proved amounts to saying that there are no hillocks or humps on the surface; there can be only cavities or pockets.

In the remainder of this section we shall assume that $f = f(p,q)$; i.e., the integrand function does not involve x, y, z explicitly. In addition to the conditions imposed at the beginning of this section, we shall also suppose that the second derivatives of f satisfy a uniform Hölder condition in each bounded portion of (p,q) -space. Consider the circle $C(x,y;r)$ with arclength s , $0 \leq s \leq 2\pi r$, and let $z(s)$ be of class $C^{(2)}$ in the closed interval $(0, 2\pi r)$. Then $z(s)$ defines a space curve C

whose projection on the xy -plane is $C(x,y;r)$; moreover, as will be proved at the end of this section, $z(s)$ satisfies the three-point condition [9, p. 49]. That is, if P, P_1, P_2 are any three distinct points on C , and if $\bar{\Phi}(P, P_1, P_2)$ denote the positive acute angle between the z -axis and the normal to the plane determined by P, P_1, P_2 then $\cos \bar{\Phi}$ is bounded away from zero for all possible distinct positions of P, P_1, P_2 on C . Hence, by an existence theorem of Haar [4, p. 19], there exists a Lipschitzian function $W(x,y)$ defined in $\bar{D}(x,y;r)$ which takes on the values $z(s)$ on $C(x,y;r)$ and which minimizes the integral I in the class of all Lipschitzian functions $z(x,y)$ having the values $z(s)$ on $C(x,y;r)$. Haar's theorem really applies to a broader situation than the one proposed here, but we shall not need his theorem in all its generality. It then turns out, by a theorem due to Morrey [7, pp. 158-160], that the minimizing function $W(x,y)$ is of class $C^{(2)}$, and the second order partial derivatives of $W(x,y)$ satisfy a Hölder condition in each closed subregion of $D(x,y;r)$.

THEOREM 5:2. Suppose $f(p,q)$ is positive regular and satisfies the differentiability conditions described above. Suppose, moreover, that $Z(x,y)$ is admissible (m), continuous in $\bar{R}$, of class $C^{(2)}$ in R , and that in every disc $\bar{D}(x,y;r)$ contained in R , $Z(x,y)$ is dominated by the minimizing function $W(x,y)$ which coincides with Z on $C(x,y;r)$. Then $I_1[\eta] \geq 0$ for every non-positive η in the class $\mathcal{N}^n$.

Suppose first that there exists an open set S , comprised with its frontier in R , and an η in $\mathcal{N}_S^n$ such that $I_1[\eta] < 0$. Then, by a procedure now familiar, we find that

$$\iint_R \eta \left(-\frac{\partial}{\partial x} f_p + \frac{\partial}{\partial y} f_q \right) dx dy > 0.$$

In view of our hypothesis on the differentiability of $Z(x, y)$ the partial derivatives written exist and are continuous. Hence, there is an open subset A of R on which $\partial f_p / \partial x + \partial f_q / \partial y < 0$. Select a circle $C(x_0, y_0; r)$ contained with its interior $D(x_0, y_0; r)$ in A . By the corollary to Theorem 4:1, it is seen that

$$\iint_{D(x_0, y_0; r)} (\xi x f_p + \eta y f_q) dx dy > 0$$

for every ξ non-negative and not equivalent to zero in $\mathcal{N}_{D(x_0, y_0; r)}^n$. Then, by (5:2) the function Z renders the integral

$$\iint_{D(x_0, y_0; r)} f(z_x, z_y) dx dy$$

a proper absolute minimum with respect to corresponding non-negative admissible variations. Consequently, by Theorem 4:3, by the comments made in this section on that theorem, and by the above remarks on the existence of minimizing functions, there exists a function W coinciding with Z on $C(x_0, y_0; r)$, minimizing the integral with respect to admissible variations (m) of arbitrary sign, and dominated by Z in $D(x_0, y_0; r)$; furthermore $Z \neq W$ there. For if it were true that $Z \equiv W$ in $D(x_0, y_0; r)$, then the Euler equation, $\partial f_p / \partial x + \partial f_q / \partial y = 0$, would hold along Z in $D(x_0, y_0; r)$, and this is contrary to the inequality deduced for the Euler expression in A . But then the hypothesis that $Z \leq W$ in $D(x_0, y_0; r)$ is contradicted. And now, to prove the required result for the case where S' coincides wholly or in part with R' , one may proceed as in the proof of Lemma 4:2.

From the foregoing results we deduce immediately the following theorem.

THEOREM 5:3. If the function $Z(x, y)$ satisfies the conditions of Theorem 5:2, then it renders the integral I a proper absolute minimum with respect to corresponding non-positive

admissible variations.

It is to be remarked that for integrand function $f(p,q)$ and Lipschitzian functions $Z(x,y)$ one may, by the "method of variation of the independent variable" employed by Haar [3, pp. 144-146] and Radó [8], deduce inequalities analogous to certain equations which are auxiliary to the Haar equation. For the case of the Haar equation these auxiliary equations are of paramount importance. For the Haar inequality, however, the author has been unable to find significant consequences of the associated inequalities, and for brevity they will be omitted here.

We shall now digress briefly into the realm of differential geometry in order to establish the validity of the three-point condition under the assumptions made on page 30 and to prove that the curve there described has an osculating plane at each point.

LEMMA 5:1. On the circle $C(x,y;r)$ with arclength s , let $z(s)$ be a periodic function of class $C^{(a)}$, with period $2\pi r$. Then $z(s)$ defines a space curve Γ satisfying the three-point condition and having an osculating plane at each point.

It obviously suffices to consider the case where $C(x,y;r)$ is the unit circle $C = C(0,0;1)$ in which event the parametric equations of the space curve may be written

$$\Gamma : x = \cos \theta, \quad y = \sin \theta, \quad z = z(\theta),$$

where $z(\theta)$ is of class $C^{(a)}$ and is periodic of period 2π . Let P , P_1 , and P_2 be three distinct points on Γ . As the angle $\bar{Q}(P, P_1, P_2)$ defined on page 31 remains invariant under rotation of the x, y axes, we may always assume that these axes are so rotated that the x -axis passes through the projection of P on C , and P corresponds to $\theta = 0$. Let θ_1 and θ_2 be parameter values determining P_1 and P_2 , respectively. The equation of the plane

through P, P_1, P_2 is then

$$\begin{vmatrix} X-1 & Y & Z-z(0) \\ \cos \theta_1 - 1 & \sin \theta_1 & z(\theta_1) - z(0) \\ \cos \theta_2 - 1 & \sin \theta_2 & z(\theta_2) - z(0) \end{vmatrix} = 0$$

Letting $a(\theta_1, \theta_2)$, $b(\theta_1, \theta_2)$, $c(\theta_1, \theta_2)$ denote, respectively, the cofactors of the elements in the first row of the determinant, we have

$$(5:3) \quad \cos \bar{\Phi}(P, P_1, P_2) = \frac{c}{(a^2 + b^2 + c^2)^{1/2}}.$$

It is easily shown that

$$c = 4 \sin \frac{\theta_2 - \theta_1}{2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}.$$

We now proceed to prove that (5:3) may be written

$$(5:4) \quad \cos \bar{\Phi}(P, P_1, P_2) = c'(\theta_1, \theta_2)/k(\theta_1, \theta_2),$$

where $k(\theta_1, \theta_2)$ is bounded and

$$c'(\theta_1, \theta_2) = \left[\sin \frac{\theta_2 - \theta_1}{2} \right] \left[\sin \frac{\theta_1}{2} \right] \left[\sin \frac{\theta_2}{2} \right].$$

Using the Maclaurin expansion for $\cos \theta$, and defining

$$v(\theta) \equiv \frac{z(\theta) - z(0)}{\theta}, \quad v(0) \equiv \lim_{\theta \rightarrow 0} \frac{z(\theta) - z(0)}{\theta} = z'(0),$$

we may obtain, after suitable manipulations on the rows of b ,

$$b = 1/2 (\theta_2 - \theta_1) \theta_1 \theta_2 b'(\theta_1, \theta_2),$$

where

$$b'(\theta_1, \theta_2) = - \begin{vmatrix} -\theta_1 + g(\theta_1) & \frac{z(\theta_1) - z(0)}{\theta_1} \\ -1 + h(\theta_1, \theta_2) & \frac{v(\theta_2) - v(\theta_1)}{\theta_2 - \theta_1} \end{vmatrix}.$$

In this determinant, $g(\theta_1)$ and $h(\theta_1, \theta_2)$ are convergent power series containing no terms of degree lower than the second. We proceed in the same manner to show that

$$a = 1/2 (\theta_2 - \theta_1) \theta_1 \theta_2 a'(\theta_1, \theta_2),$$

where

$$a'(\theta_1, \theta_2) = \begin{vmatrix} 2 + g_0(\theta_1) & \frac{z(\theta_1) - z(0)}{\theta_1} \\ h_0(\theta_1, \theta_2) & \frac{v(\theta_2) - v(\theta_1)}{\theta_2 - \theta_1} \end{vmatrix},$$

and g_0, h_0 are convergent power series containing no terms of degree lower than the first.

We shall be certain that $a'(\theta_1, \theta_2)$ and $b'(\theta_1, \theta_2)$ are uniformly bounded as soon as the boundedness of $|v(\theta_2) - v(\theta_1)|/(\theta_2 - \theta_1)$ has been verified. Now $v(\theta)$ is continuous and may be shown to possess a derivative everywhere; in fact, for $\theta \neq 0$, we have,

$$v'(\theta) = \frac{\theta z'(\theta) - z(\theta) + z(0)}{\theta^2}.$$

Expressing $z(0)$ by Taylor's expansion with remainder about the point θ , we find this equality to be equivalent to

$$v'(\theta) = z''(\bar{\theta})/2, \quad 0 < |\bar{\theta}| < |\theta|.$$

Also, by the definition of $v(\theta)$, we obtain

$$\frac{v(\theta) - v(0)}{\theta} = \frac{z(\theta) - z(0) - \theta z'(0)}{\theta^2}.$$

Again expanding $z(\theta)$, this time about the point $\theta = 0$, it is seen that

$$\frac{v(\theta) - v(0)}{\theta} = z''(\theta^*)/2, \quad 0 < |\theta^*| < |\theta|.$$

Therefore, by the continuity of z'' , $v'(0)$ exists and is equal to $z''(0)/2$. Hence $v'(\theta)$ exists everywhere on the given circle and is uniformly bounded by the finite upper bound of $z''(\theta)$. Then by the mean value theorem, $[v(\theta_2) - v(\theta_1)]/(\theta_2 - \theta_1) = v'(\bar{\theta})$, where $\bar{\theta}$ is an intermediate value of θ , so that a' and b' are clearly bounded. This result, together with (5:3), implies (5:4) where $k(\theta_1, \theta_2) = (a'^2 + b'^2 + c'^2)^{1/2}$ is bounded. It should be remarked that a' , b' , c' are proportional to the direction cosines of the normal to the plane through P , P_1 , P_2 .

Let ϵ be a small positive number, and restrict θ_1, θ_2 to be in the closed interval $(-\pi, \pi - \epsilon)$. Then none of the three factors of $c'(\theta_1, \theta_2)$ can become zero; indeed, each factor assumes a positive minimum. Consequently, since $k(\theta_1, \theta_2)$ is bounded, the product in (5:4) is bounded away from zero. Let the greatest lower bound of $\cos \Phi(P, P_1, P_2)$ be denoted by m_1 , for θ_1, θ_2 restricted as above. Let m_2 and m_3 have similar meanings corresponding to the respective closed intervals $(-\pi + \epsilon, \pi)$ and $(\pi - \epsilon, \pi + \epsilon)$. Every possible triple of distinct points P, P_1, P_2 on Γ corresponds to a pair of values θ_1, θ_2 lying in at least one of the three intervals indicated above. Hence, for all possible triples of distinct points P, P_1, P_2 , the function $\cos \Phi(P, P_1, P_2)$ is bounded away from zero by the smallest of the numbers m_1, m_2, m_3 , and the three-point condition is therefore satisfied.

The quantities θ_1, θ_2 will now be confined to an interval $(-\epsilon, \epsilon)$, where ϵ is small, and it will be proved that the direction numbers $a'(\theta_1, \theta_2)$, $b'(\theta_1, \theta_2)$, $c'(\theta_1, \theta_2)$ approach limits as P_1, P_2 approach P independently along Γ . Now in view of the properties of $v(\theta)$ inferred in the foregoing proof, we see that

$$\frac{v(\theta_2) - v(\theta_1)}{\theta_2 - \theta_1} = v'(\bar{\theta}) = z''(\theta^*)/2,$$

where $\bar{\theta} = \theta_1 + a(\theta_2 - \theta_1)$ for some $0 < a < 1$, and $0 < |\theta^*| < |\bar{\theta}|$.

Then $|\theta^*| \leq \max \{|\theta_1|, |\theta_2|\}$. Hence, it is easily seen from the continuity of z'' that

$$\lim_{\substack{\theta_1 \rightarrow 0 \\ \theta_2 \rightarrow 0}} \frac{v(\theta_2) - v(\theta_1)}{\theta_2 - \theta_1} = z''(0)/2.$$

It now follows readily that

$$\lim_{\substack{\theta_1 \rightarrow 0 \\ \theta_2 \rightarrow 0}} a'(\theta_1, \theta_2) = z''(0), \quad \lim_{\substack{\theta_1 \rightarrow 0 \\ \theta_2 \rightarrow 0}} b'(\theta_1, \theta_2) = z'(0), \quad \lim_{\substack{\theta_1 \rightarrow 0 \\ \theta_2 \rightarrow 0}} c'(\theta_1, \theta_2) = 1.$$

Thus, the plane determined by the points P, P_1, P_2 approaches a limiting position as P_1, P_2 approach P along the curve, and the limiting plane is by definition the osculating plane to Γ at P . But P is an arbitrary point on Γ , so that we have proved that Γ has an osculating plane at each point.

6. Subharmonic functions. In this section we shall consider, as a particular case of the class of integrand functions treated in Section 5, the function $f(p, q) = p^2 + q^2$, the integrand of the Dirichlet integral. The problem treated here is clearly positive regular. If $Z(x, y)$ is single valued and measurable in $\bar{R}$, and if the partial derivatives Z_x, Z_y exist almost everywhere and belong to L^2 in R , then Z is admissible (2). The class of corresponding admissible variations in this case is composed of functions $\eta(x, y)$, continuous in $\bar{R}$, a.c.s. in R , vanishing on R' , and whose first order partial derivatives belong to L^2 in R . This section will be concerned partly, but not entirely, with functions and variations which are admissible (2); for simplicity, in this section we shall say " Z is admissible" instead of the more exact phrase " Z is admissible (2)".

Before undertaking the subject of subharmonic functions it will be necessary to make a few preliminary assertions regarding harmonic functions. Let $U(x,y)$ be harmonic on an open set S and continuous in $\bar{S}$. Then $U(x,y)$ satisfies the Haar integral equation; i.e.,

$$\int_C U_x dy - U_y dx = 0$$

for every simply closed rectifiable curve C contained with its interior in S . McShane [6] has proved that U renders the Dirichlet integral

$$I[z;S] = \iint_S (z_x^2 + z_y^2) dx dy$$

an absolute minimum in the class of functions $z(x,y)$ coinciding with U on S' , continuous in $\bar{S}$, and a.c.s. in S . The following possibility should be observed. We may have $I[U;S] = \infty$; in such a case, on account of the minimizing property of U , the class of admissible functions continuous in $\bar{S}$, a.c.s. in S , and coinciding with $U(x,y)$ on S' will be vacuous. We shall come back to this point later in our discussion.

We shall employ the following definition of subharmonic function which, except for the final statement, is the same as that given by Radó [11, p. 1].

DEFINITION 6:1. A function $u(x,y)$ defined in a connected region R is subharmonic in R if it satisfies the following conditions:

- (1) $-\infty \leq u < +\infty$ in R ;
- (ii) u is not identically equal to $-\infty$ in R ;
- (iii) u is upper semi-continuous in R ;
- (iv) if S is any open region such that $\bar{S}$ is contained in R , and if $U(x,y)$ is harmonic in S , continuous in $\bar{S}$, and $U \geq u$ on

S' , then $U \geq u$ in S .

If the region R is not connected, then $u(x,y)$ is subharmonic in R in case it is subharmonic in every connected subregion of R .

A function u which is subharmonic in R is summable on every measurable set comprised with its frontier in R [11, p.4]. Furthermore, u is summable as a function of the polar angle on every circle $C(x_0, y_0; r)$ contained with its interior in R [11, p. 4]. A fundamental property of subharmonic functions is that

$$u(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos \theta, y_0 + r \sin \theta) d\theta$$

for every (x_0, y_0) in R and r such that $\bar{D}(x_0, y_0; r)$ is in R [11, pp. 3,4]. Conversely, a function satisfying this inequality in R and also (i), (ii) and (iii) of Definition 6:1 is certainly subharmonic in R [11, p. 7]. Another important inequality satisfied by subharmonic functions, and derived from the above, is

$$u(x_0, y_0) \leq \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r u(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) \rho d\rho d\theta,$$

for every (x_0, y_0) in R and r such that $\bar{D}(x_0, y_0; r)$ is in R [11, pp. 3,4]. Again, the validity of this inequality in R , together with (i), (ii) and (iii) of Definition 6:1, is sufficient to characterize a function as subharmonic in R [11, p. 7].

F. Riesz has shown [13, part II] that if $u(x,y)$ is an arbitrary subharmonic function in the region R , then in any open set G contained with its frontier in R , u may be represented as the sum of the potential of a negative mass distribution and a harmonic function. A simplified development of this representation due to G. C. Evans [1, part I, p. 237] is reviewed by Radó [11, pp. 40-42]. In consequence of this result, it may be shown [11, p. 45] that in G the subharmonic function u is an absolutely

continuous function of y for almost every x , and an absolutely continuous function of x for almost every y ; moreover, its partial derivatives are summable in G . It is to be observed that u need not be admissible, so that the possibility exists that $I[u;R] = \infty$.

It is well known that the function defined at each point to be equal to the integral mean of a subharmonic function over a circular disc of fixed radius is also subharmonic within its range of definition [11, pp. 10, 11]. But this property is also valid if the integral means be taken over square regions [10, p. 347]; thus the function $u^{(h)}(x,y)$ is subharmonic in its region of definition N_h . And now, by the remarks of the preceding paragraph and Property 3:7 we have that

$$(6:1) \quad \frac{\partial}{\partial x} u^{(h)} = \left(\frac{\partial u}{\partial x} \right)^{(h)},$$

$$(6:2) \quad \frac{\partial}{\partial y} u^{(h)} = \left(\frac{\partial u}{\partial y} \right)^{(h)}$$

in N_h . Therefore, if u is subharmonic in R , then $u^{(h)}$ is of class C^1 in N_h .

Having made the foregoing observations, we are now prepared to present a number of propositions which correspond to theorems and properties already obtained in Sections 4 and 5. We shall restrict ourselves, for the time being, to admissible functions and variations. We notice that for admissible functions and variations $w(x,y)$ and $\eta(x,y)$, respectively, the first variation is

$$I_1[\eta] = \iint_R (\eta_x w_x + \eta_y w_y) dx dy,$$

while the Haar inequality is

$$\int_Q w_x dy - w_y dx \geq 0.$$

Corresponding to Theorem 4:1 we have that if $I_1[\eta] \geq 0$ for all non-positive admissible variations, then the Haar inequality holds for almost all squares of R , and conversely. Inasmuch as the proposition corresponding to Theorem 5:1 is to be referred to later on in the proof of a somewhat stronger theorem, we shall write it as a lemma.

LEMMA 6:1. The non-negativeness of the first variation, or the validity of the Haar inequality for almost all squares of R , implies that an admissible w affords the Dirichlet integral a proper minimum in the class of functions $w + \eta$, where η is a non-positive admissible variation.

We may also apply Theorem 4:3 to our present considerations. However, in view of remarks made in Section 5 on that theorem, its statement may be simplified. We again begin with an admissible surface $w(x,y)$ which minimizes the Dirichlet integral with respect to non-positive admissible variations. Suppose, furthermore, that $w(x,y)$ is continuous in $\bar{R}$ and a.c.s. in R . Then, if S is any open subset of R for which there exists a function $U(x,y)$ harmonic in S , continuous in $\bar{S}$, and $U \geq w$ on S' , it follows, as a result of the minimizing property of harmonic functions indicated earlier, that $U \geq w$ in S . Hence, by Definition 6:1, $w(x,y)$ is subharmonic in R .

It was stated in the introductory paragraph of Section 5 that for the Dirichlet integral the theorem corresponding to Theorems 5:2 and 5:3 could be proved under weaker hypotheses than those there required. The theorem to be proved is the following:

THEOREM 6:1. Let $u(x,y)$ be a function subharmonic in R . Then, for almost all squares Q in R ,

$$\int_Q u_x dy - u_y dx \geq 0.$$

Let $C(x_0, y_0; r)$ be any circle contained with its interior $D(x_0, y_0; r)$ in N_h , and let (x, y) be the variable point $x = x_0 + \rho \cos \theta$, $y = y_0 + \rho \sin \theta$, where $0 \leq \theta < 2\pi$, and $0 < \rho \leq r$. Since $u^{(h)}(x, y)$ has continuous partial derivatives in N_h , we have,

$$(6:3) \quad \begin{aligned} & \frac{\partial}{\partial \rho} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) \\ &= \frac{\partial}{\partial x} u^{(h)}(x, y) \cos \theta + \frac{\partial}{\partial y} u^{(h)}(x, y) \sin \theta \end{aligned}$$

in the disc $D(x_0, y_0; r)$. It now follows that for $0 < \rho \leq r$,

$$(6:4) \quad \begin{aligned} & \frac{d}{d\rho} \frac{1}{2\pi} \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta \\ &= \frac{1}{2\pi} \int_{C(x_0, y_0; \rho)} u_x^{(h)} dy - u_y^{(h)} dx \geq 0, \end{aligned}$$

since the integral mean

$$\frac{1}{2\pi} \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta$$

is a non-decreasing function of ρ [11, p. 8]. Now $u_x^{(h)}$ and $u_y^{(h)}$ are absolutely continuous in the sense of Tonelli in N_h by Property 3:8, and therefore the partial derivatives $\partial u_x^{(h)} / \partial x$, $\partial u_y^{(h)} / \partial y$ are summable in arbitrary closed subsets of N_h by Property 3:9. Under these conditions, it is permissible to apply Green's lemma to the line integral appearing in (6:4). We then obtain

$$(6:5) \quad \frac{1}{\pi \rho^2} \iint_{D(x_0, y_0; \rho)} \left(\frac{\partial}{\partial x} u_x^{(h)} + \frac{\partial}{\partial y} u_y^{(h)} \right) dx dy \geq 0, \quad (0 < \rho \leq r).$$

But the left-hand member of (6:5) is the integral mean of $\partial u_x^{(h)} / \partial x + \partial u_y^{(h)} / \partial y$ on $D(x_0, y_0; \rho)$. Such integral means

taken over discs have essentially the same properties as those taken over square regions. In particular, Property 3:2 holds for means of this type. We may obtain an inequality similar to (6:5) for each point (x_0, y_0) in N_h . Passing to the limit in (6:5) as ρ approaches zero, we find that for every $h > 0$, the inequality

$$(6:6) \quad \frac{\partial}{\partial x} u_x^{(h)} + \frac{\partial}{\partial y} u_y^{(h)} \geq 0$$

holds almost everywhere in N_h . Inequality (6:6) may be written almost everywhere as

$$(6:7) \quad \frac{1}{4h^2} \int_Q u_x dy - u_y dx \geq 0, \quad [h > 0; Q = Q(x, y; h) \subset N_h].$$

We thus see that in accordance with Definition 4:1 the inequality (6:7) holds for almost all squares in R . This completes the proof.

It has not been assumed in the above proof that the subharmonic function $u(x, y)$ be admissible; i.e., it has not been required that u_x and u_y belong to L^2 . As for the converse, if u is actually continuous in $\bar{R}$, a.c.s. in R , admissible, and satisfies the Haar inequality for almost all squares of R , then by the propositions on page 41 it results that u is subharmonic in R . But we may deduce a converse of Theorem 6:1 under still weaker hypotheses. This will be done in the next theorem. We first introduce the definition of almost subharmonic function in a region. Except for the last statement, it is the same as that presented by Radó [11, p. 20].

DEFINITION 6:2. A function u is almost subharmonic in a connected region R if it satisfies the following conditions:

(1) u is summable on every measurable set contained together with its frontier in R ;

(ii) almost everywhere in R the inequality

$$u(x_0, y_0) \leq \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} u(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta$$

holds for every r such that $\bar{D}(x_0, y_0; r)$ is in R.

If the region R is not connected, then $u(x, y)$ is almost subharmonic in R in case it is almost subharmonic in every connected subregion of R.

It results from this definition that a function u is almost subharmonic in R if and only if there exists in R a subharmonic function u^* such that $u = u^*$ almost everywhere in R [11, pp. 20, 21]. Hence, we see that many of the properties herein developed for subharmonic functions may also be ascribed to almost subharmonic functions.

THEOREM 6:2. Suppose that $u(x, y)$ is summable in every open set comprised with its frontier in R together with its first order partial derivatives, and that in R the function u is absolutely continuous in x for almost every y and absolutely continuous in y for almost every x . If, furthermore,

$$\int_Q u_x dy - u_y dx \geq 0$$

for almost all squares of R , then u is almost subharmonic in R . If, in addition, u is continuous in R , i.e., if u is a.c.s. in R , then it is subharmonic in R .

The proof, for the most part, amounts to reversing the procedure in the proof of Theorem 6:1. By Property 3:7, it follows from our hypothesis that $\partial u_x^{(h)} / \partial x + \partial u_y^{(h)} / \partial y \geq 0$ almost everywhere in N_h . By Property 3:9, the partial derivatives in the left-hand member are summable on closed subsets of N_h . If $\bar{D}(x_0, y_0; r)$ is in R , we thus have

$$\iint_{D(x_0, y_0; \rho)} \left(\frac{\partial}{\partial x} u^{(h)} + \frac{\partial}{\partial y} u^{(h)} \right) dx dy \geq 0, \quad (0 < \rho \leq r).$$

By Green's lemma and Property 3:7 it follows that

$$\int_{C(x_0, y_0; \rho)} \frac{\partial}{\partial x} u^{(h)} dy - \frac{\partial}{\partial y} u^{(h)} dx \geq 0, \quad (0 < \rho \leq r),$$

and the argument which led to equation (6:4) now implies

$$\frac{d}{d\rho} \frac{1}{2\pi} \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta \geq 0, \quad (0 < \rho \leq r).$$

Hence the integral mean

$$\frac{1}{2\pi} \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta$$

is a monotone non-decreasing function of ρ for each (x_0, y_0) in N_h . By the mean value theorem for integrals, and by the continuity of $u^{(h)}$, we now deduce that the inequality

$$(6:8) \quad u^{(h)}(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta, \quad (0 < \rho \leq r),$$

holds for all (x_0, y_0) in N_h and r such that $\bar{D}(x_0, y_0; r)$ is in N_h .

To show that u is almost subharmonic, integrate both sides of the above expression between the limits 0 and r . Thus,

$$\begin{aligned} u^{(h)}(x_0, y_0) &= \frac{2}{r^2} \int_0^r u^{(h)}(x_0, y_0) \rho d\rho \\ &\leq \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) \rho d\rho d\theta. \end{aligned}$$

And now, by Properties 3:2 and 3:5, we finally conclude that

$$u(x_0, y_0) \leq \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} u(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) \rho d\rho d\theta$$

almost everywhere in R . Hence, $u(x, y)$ is almost subharmonic in R .

If u had been taken to be continuous, then passing to the limit in (6:8) it would follow that

$$u(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta$$

everywhere in R ; that is, u would be subharmonic in R [11, p. 7].

If $u(x, y)$ is an admissible subharmonic function, then by Theorem 6:1 and Lemma 6:1 the function u affords the Dirichlet integral an absolute minimum with respect to non-positive admissible variations. We shall now prove a stronger theorem which is analogous to the theorem of McShane referred to in an early paragraph of this section.

THEOREM 6:3. If $u(x, y)$ is a subharmonic function in R , then u renders the Dirichlet integral an absolute minimum in the class of functions $z = u + \eta$ where η is continuous in $\bar{R}$, a.c.s. in R , equal to zero on R' , and non-positive in R .

We shall first prove the theorem for the case where $\eta \equiv 0$ in a neighborhood of the frontier R' . Indeed, let S be any open set comprised together with its frontier in R , and suppose $\eta \equiv 0$ in $\bar{R} - S$. Select any open set S_1 such that $R \supset \bar{S}_1$ and $S_1 \supset \bar{S}$. Let d be the minimum of the distances of points of S_1' from points of $R' + S'$. Then $z^{(h)}$ and $u^{(h)}$ are defined in $\bar{S}_1$ for $h < d/\sqrt{2}$. Also, $z^{(h)} = u^{(h)}$ on S_1' and $z^{(h)} \leq u^{(h)}$ in S_1 . Moreover, $z^{(h)}$ and $u^{(h)}$ are of class C' in $\bar{S}_1$, and $u^{(h)}$ is subharmonic in S_1 . Then, by Theorem 6:1 and Lemma 6:1,

$$(6:9) \quad I[u^{(h)}; S_1] \leq I[z^{(h)}; S_1].$$

If $I[z; R] = \infty$, then the result of the theorem is trivial. Let us assume, therefore, that $I[z; R] < \infty$, so that z_x and z_y belong to L^2 . Then, we easily show by Schwarz's inequality that

$$(6:10) \quad (z_x^{(h)})^2 \leq (z_x^2)^{(h)}.$$

Now, by Fatou's lemma [14, p. 29], Properties 3:2 and 3:7, and by (6:9) and (6:10), we obtain the sequence of inequalities

$$\begin{aligned}
 (6:11) \quad I[u;S_1] &\leq \liminf_{h \rightarrow 0} I[u^{(h)};S_1] \leq \liminf_{h \rightarrow 0} I[z^{(h)};S_1] \\
 &\leq \lim_{h \rightarrow 0} \iint_{S_1} (z_x^2 + z_y^2)^{(h)} dx dy = I[z;S_1].
 \end{aligned}$$

At each point of $R - S$ the function $z - u$ is zero, the maximum of its values in R . Hence, at every point of $R - S$ where z_x , z_y , u_x , u_y exist, we must have $z_x = u_x$, $z_y = u_y$. This occurs almost everywhere in $R - S$. Therefore, $I[u;R - S_1] = I[z;R - S_1]$ and by (6:11),

$$(6:12) \quad I[u;R] \leq I[z;R].$$

If we had $I[u;R] = \infty$, then clearly we must have $I[z;R] = \infty$; otherwise the sequence of inequalities in (6:11) would lead to a contradiction in (6:12).

The remainder of the argument consists in extending the result just obtained to any function $z = u + \eta$ satisfying the conditions of the theorem. For harmonic functions McShane [6, p. 595] performs this extension in a very efficient and concise manner, so the particulars will not be repeated here.

Recapitulating some of the results of this section, we see that there is an association between subharmonic functions and the calculus of variations. For continuous subharmonic functions a number of characterizing properties may be stated which have interest from the point of view of the calculus of variations. A function $u(x,y)$ which is continuous in R has been found to be subharmonic in R if and only if it satisfies any one of the following equivalent conditions:

- (1) it fulfills condition (iv) of Definition 6:1;

(ii) it is a.c.s. in R , its first order partial derivatives are summable in every closed subset of R , and

$$\int_Q u_x dy - u_y dx \geq 0 \text{ for almost all squares of } R;$$

(iii) it is a.c.s. in R and renders the Dirichlet integral a proper minimum in the class of functions $z = u + \eta$, where η is continuous in $\bar{R}$, a.c.s. in R , equal to zero on R' , and non-positive in R .

Actually, the necessity of the last property was proved in Theorem 6:3 under somewhat weaker assumptions on u and z than those stated here.

7. Association with the problem of Plateau in non-parametric form. The theory of Section 5 may be applied without difficulty to the integrand $f(p,q) = (1 + p^2 + q^2)^{1/2}$. It is clear that this problem is positive regular, for $f_{pp} = (1 + q^2)/f^3$, $f_{qq} = (1 + p^2)/f^3$, and $f_{pp}f_{qq} - f_{pq}^2 = 1/f^2$.

Let R be a Jordan region; in particular, for simplicity we shall take R as a rectangle $R: a < x < b; c < y < d$. If $z = Z(x,y)$ is continuous in $\bar{R}$ and has a finite area in the sense of Lebesgue [9, p. 6], then this area is given by the integral

$$I = \iint_R (1 + z_x^2 + z_y^2)^{1/2} dx dy$$

if and only if z is A.C.T. [9, p. 10]*. If $Z(x,y)$ renders the area integral a minimum in the class of all Lipschitzian functions defined in $\bar{R}$ and assuming the same boundary values, then $Z(x,y)$ is analytic and, for these boundary values, is a solution of the Problem of Plateau; i.e., $Z(x,y)$ satisfies the partial differential equation

* For the case of the general Jordan region, see E. J. MoShane, Integrals over surfaces in parametric form. Annals of Mathematics (2) vol. 34 (1933), pp. 815-838; in particular, p. 836.

$$(7:1) \quad (1 + z_x^2)z_{yy} - 2z_xz_yz_{xy} + (1 + z_y^2)z_{xx} = 0$$

in R [9, p. 61]. Except for a multiplicative factor $1/f^3$, equation (7:1) is the Euler equation for this calculus of variations problem and is equivalent to the statement that the mean curvature of Z is everywhere zero. The surface $Z(x,y)$ is then a minimal surface.

In general, if S is the interior of a convex Jordan curve S' , then there exists in S a minimal surface of the form $z = Z_0(x,y)$ taking on prescribed continuous boundary values on S' , and $z = Z_0(x,y)$ is a minimal surface whose area is actually a minimum with respect to all continuous surfaces of the type of the circular disc, having the same boundary values on S' [9, p. 101]. This fact is used to obtain the concluding inequality (7:2) of this section.

In view of the nature of the integrand function $f(p,q)$ it is seen that Z is admissible (∞) if it is single valued and measurable in $\bar{R}$, and if its first order partial derivatives are summable in R . The corresponding admissible variations are functions $\eta(x,y)$, continuous in $\bar{R}$, a.c.s. in R , vanishing on R' , and whose first order partial derivatives are summable in R . In particular, it is to be noted that admissible variations are A.C.T. in R .

We might observe that the first variation reduces for the case here treated to

$$I_1[\eta] = \iint_R \frac{1}{f(Z_x, Z_y)} (\eta_x Z_x + \eta_y Z_y) dx dy,$$

and the Haar inequality becomes

$$\int_Q \frac{1}{f(Z_x, Z_y)} (Z_x dy - Z_y dx) \geq 0.$$

Plainly, the results of earlier sections may be stated so as to be applicable in the present discussion. Corresponding to

Theorem 4:1 we conclude that $I_1[\eta] \geq 0$ for non-positive admissible variations if and only if the Haar inequality holds. Moreover, it follows at once from Theorem 5:2 that if $I_1[\eta] \geq 0$ for non-positive admissible variations, or if the Haar inequality holds in R , then Z minimizes the integral I with respect to non-positive admissible variations.

As a consequence of Theorem 5:2 and the opening part of this section we note that if an admissible Z is continuous in $\bar{R}$, of class $C^{(2)}$ in R , and if in every disc $D(x,y;r)$ in R , we have $Z \leq W$, where $W(x,y)$ is the minimal surface coinciding with Z on $C(x,y;r)$, then $I_1[\eta] \geq 0$ for all non-positive admissible variations.

It is now evident that if the surfaces under consideration be continuous in $\bar{R}$ and of class $C^{(2)}$ in R , then the hypotheses and conclusions appearing in the underlined propositions of this section are equivalent. By the method employed in the proof of Theorem 5:2 we may then derive an inequality analogous to the Euler equation (7:1), namely,

$$(7:2) \quad (1 + Z_x^2)Z_{yy} - 2Z_x Z_y Z_{xy} + (1 + Z_y^2)Z_{xx} \geq 0,$$

which must hold at all points of R . This implies that the mean curvature of Z is non-negative throughout R .

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